Drift- or Fluctuation-Induced Ordering and Self-Organization in Driven Many-Particle Systems

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Abstract

According to empirical observations, some pattern formation phenomena in driven many-particle systems are more pronounced in the presence of a certain noise level. We investigate this phenomenon of fluctuation-driven ordering with a cellular automaton model of interactive motion in space and find an optimal noise strength, while order breaks down at high(er) fluctuation levels. Additionally, we discuss the phenomenon of noise- and drift-induced self-organization in systems that would show disorder in the absence of fluctuations. In the future, related studies may have applications to the control of many-particle systems such as the efficient separation of particles. The rather general formulation of our model in the spirit of game theory may allow to shed some light on several different kinds of noise-induced ordering phenomena observed in physical, chemical, biological, and socio-economic systems (e.g., attractive and repulsive agglomeration, or segregation).

Noise-related phenomena can be quite surprising and exciting. Therefore, they have attracted the interest of many researchers. For example, we mention stochastic resonance [1], structural (in)stability [2, 3], noise-driven motion ('Brownian motors') [4, 5, 6], and "freezing by heating" [7]. The approach proposed in this contribution differs from these phenomena. Moreover, since both, the initial conditions and the interaction strengths in our model are assumed to be independent of the position in space, the fluctuation-induced self-organization discussed later on should be distinguished from so-called "noise-induced transitions" in systems with multiplicative noise as well, where we have a space-dependent diffusion coefficient which can induce a transition [8]. Although our model is related with diffusive processes, it is also different from reaction-diffusion systems that can show fluctuation-induced self-organization phenomena known as Turing patterns [9, 10, 11, 12, 13, 14].

The main goals of this contribution are (i) to qualitatively understand the observed noise-induced ordering phenomena in certain self-driven many-particle systems and (ii) to derive and investigate a simplified mathematical model for them. Some of the properties of this model appear reminiscent of driven (e.g. shared) heterogeneous granular systems. For example, one finds stratification phenomena such as the segregation of different kinds of grains into layers [15, 16]. This mechanism explains, for example, some of the geological formations and ore concentrations in the earth. As it may be also used to separate different materials, it is interesting to ask for the most efficient separation technique including the optimal noise level. A similar segregation phenomenon of lane formation has been found in pedestrian counter-streams [17]. This is based on a reduction of the interaction strength and related with an increase in the efficiency or average speed of motion [18].

In the following, we will explore a cellular automaton model which extends the one discussed in Ref. [18] by the effects of fluctuations, drift, and asymmetric interactions. It may, for example, reflect the interactive one-dimensional motion of some driven many-particle systems in a spatial direction perpendicular to the main flow direction(s) and to the boundaries. For this purpose, let us imagine the example of pedestrian streams in a corridor with two opposite flow directions a. We subdivide the one-dimensional space into cells i comparable to the shoulder width (diameter) Δx of the pedestrians (particles, entities). Speaking in more general terms, we have A subpopulations a with N_a entities α distributed over I cells $i \in \{1, \dots, I\}$. We denote the number of entities in cell i belonging to subpopulation a by n_i^a. Moreover, we represent the kind of interaction and the interaction strength between two entities of subpopulations a and b by a constant parameter value P_{ab}. (Generalizations are easily possible.) Finally, we update the locations of the entities α according to the following rules: *1st step:* Select the entity α randomly. If α is located in cell i and belongs to subpopulation a,

determine

$$S_\alpha(j,t) = \sum_b P_{ab}\, n_j^b(t) + \xi_j^\alpha(t) \tag{1}$$

for $j = i$ and the nearest neighbors $j = i \pm 1$, where $\xi_j^\alpha(t)$ are random fluctuations uniformly distributed in the interval $[-p_a, p_a]$, so that p_a denotes the fluctuation strength. *2nd step:* Move to the neighboring cell $i \pm 1$ with probability

$$P_\alpha(i \pm 1|i;t) \propto \max\{0, S_\alpha(i \pm 1, t) - S_\alpha(i,t)\}\,. \tag{2}$$

3rd step: Repeat steps 1 and 2 until the locations of $N = \sum_a N_a$ entities were updated. *4th step:* With probability V_a^0, move all entities α of subpopulation a by one cell into the same direction. *5th step:* Return to step 1.

Formula (1) calculates the expected effect of interactions with other entities. $S_a(i,t)$ is a potential function, see Eq. (4). In the language of game theory, it can be called the *expected success*, since, according to the *proportional imitation rule* (2), an entity α moves to a neighboring cell $i \pm 1$ only if it can increase the value of S_α. The values P_{ab} may be interpreted as *payoffs* in interactions between two entities of subpopulations a and b. P_{ab} is positive for attrative, cooperative, or profitable interactions, while it is negative for repulsive, competitive, or loss-making interactions. The fourth step reflects a bias in the motion of the particles of subpopulation a, i.e. a *drift velocity*.

We have carried out various simulations with random initial and periodic boundary conditions, in order to have a translation-invariant system. Note that self-organized pattern formation in such a system implies spontaneous symmetry-breaking and a pronounced history-dependence of the resulting state. The typical solutions are dependent on the specified payoffs P_{ab}, fluctuation strengths p_a, and drift velocities V_a^0. Replacing the asynchronous (random sequential) update of steps 1 to 3 by a parallel update yields similar (but less random, i.e. "nicer looking") results. Replacing the parallel update of the velocities in step 4 by an asynchronous one induces such a high noise level that the system is often disordered.

To obtain a theoretical understanding of our simulation results, we have derived mean value equations for the densities $\rho_a(x,t) = n_i^a(t)/\Delta x$ with $x = i\Delta x$. For this purpose, we have derived a master equation and determined the drift- und fluctuation-coefficients in the usual way. By second order Taylor expansion, the resulting equations can be then approximately cast into the form of Fokker-Planck equations [19]

$$\frac{\partial \rho_a(x,t)}{\partial t} + \frac{\partial}{\partial x}\left[\rho_a(x,t) V_a(x,t)\right] = D_a \frac{\partial^2 \rho_a(x,t)}{\partial x^2} \tag{3}$$

where the diffusion coefficients D_a increase with the fluctuation strength p_a in a roughly proportional way. Moreover, D_a vanishes when p_a vanishes and $|\partial S_a(x,t)/\partial x|$ is small (as for our homogeneous initial conditions). The exact

relation for D_a is rather complicated, but not of interest, here. Finally, the drift coefficients are given by

$$V_a(x,t) = V_a^0 + \frac{\partial S_a(x,t)}{\partial x} \qquad \text{with} \qquad S_a(x,t) = \sum_b P_{ab}\, \rho_b(x,t)\,. \tag{4}$$

Accordingly, we are confronted with non-linearly coupled Burgers equations, which may show a diffusion instability. In the following, we will check whether the above mean value equations yield qualitatively meaningful results, i.e. whether correlations can be neglected. For this purpose, we will carry out a linear stability analysis and compare the theoretical phase diagram with the numerically determined one. In the case of two subpopulations $a \in \{1,2\}$ and $V_1^0 = V_2^0 = 0$, the homogeneous solution $\rho_a^0 = \overline{n_i^a}/\Delta x$ with $\overline{n_i^a} = N_a/I$ should be unstable with respect to fluctuations, if

$$\rho_1^0 P_{11} + \rho_2^0 P_{22} > D_1 + D_2 \tag{5}$$

or

$$\rho_1^0 \rho_2^0 P_{12} P_{21} > (\rho_1^0 P_{11} - D_1)(\rho_2^0 P_{22} - D_2)\,. \tag{6}$$

Let us first discuss the the symmetric case with $\rho_1^0 = \rho_2^0 = \rho$, $P_{11} = P_{22} = P$, $P_{12} = P_{21} = Q$, and vanishing diffusion $D_1 = D_2 = 0$. Then, condition (5) reduces to $2\rho P > 0$, and condition (6) becomes $Q^2 > P^2$. We can distinguish the following solutions (see Fig. 1a, for representatives see Fig. 4 in Ref. [18]): A) If $P < 0$ and $Q^2 < P^2$ [i.e. $P < 0$ and $P < Q < -P$)], a *homogeneous distribution* $\rho_a(x,t) = \rho_a^0$ over all sites in *both* subpopulations is stable with respect to small perturbations (which corresponds to *disorder*). B) If $P < 0$ (self-repulsion) and $Q < 0$ (repulsion between the subpopulations), but $Q < P$, we should find *segregation* (with a tendency that *all* sites are equally occupied, but *either* by one subpopulation *or* by the other). C) If $Q < 0$ (repulsion), but $P > 0$ (self-attraction), we expect *repulsive agglomeration* (i.e. both subpopulations should cluster at *different* sites, with empty sites in between). D) If $Q > 0$ (attraction) and $Q > -P$, we should have *attractive agglomeration* (clustering of both subpopulations at the *same* sites, with empty sites in between). Consequently, on the line $Q = (P-1)/2$ (i.e. for $P = 2Q+1$), we should cross the phase boundary between disorder and segregation at $P = -1$, the one between segregation and repulsive agglomeration at $P = 0$, and the one between repulsive and attractive agglomeration at $P = +1$. This is, in fact, confirmed by our numerical simulations (see Fig. 1b), so that we can trust the instability analysis based on the mean value equations. The reason for this is the local nature of the interactions. To characterize the different states, we have used order parameters of the form

$$\Theta(y) = \frac{1}{I}\sum_{i=1}^{I} (y_i - \overline{y_i})^2 \qquad \text{with} \qquad \overline{y_i} = \frac{1}{I}\sum_{i=1}^{I} y_i \tag{7}$$

to measure the variances of (i) $y_i = (n_i^1/\overline{n_i^1} + n_i^2/\overline{n_i^2})$ (i.e. the deviation from a homogeneous occupation of *all* sites), or (ii) $y_i = (n_i^1/\overline{n_i^1} - n_i^2/\overline{n_i^2})$ (i.e. the difference in the degree of occupation by different subpopulations). $\Theta(n^1+n^2)$ is sensitive to (attractive or repulsive) agglomeration (i.e. to clustering with empty sites in between), and $\Theta(n^1+n^1)$ recognizes, when the two subpopulations tend to use different sites (as for segregation or repulsive agglomeration).

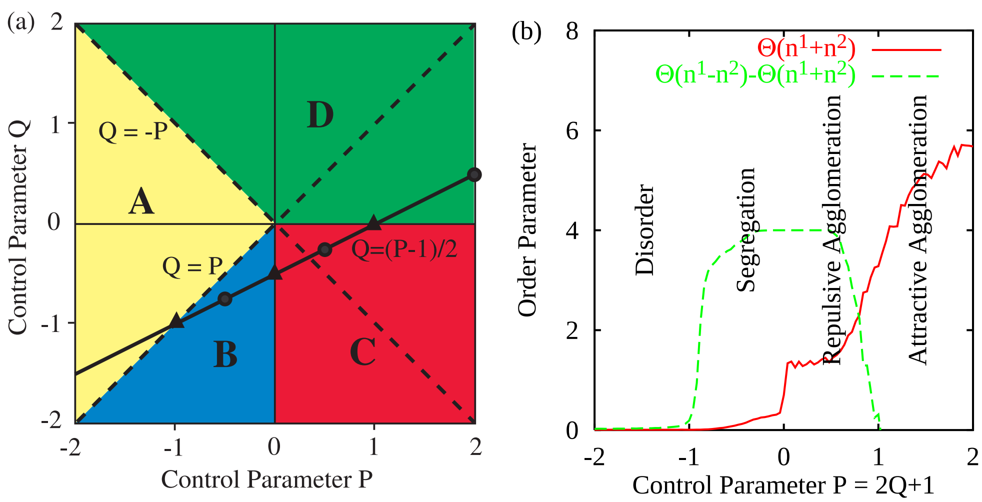

Figure 1: (a) Theoretical phase diagram of the four qualitatively different patterns resulting for the symmetric model without diffusion: A = disordered phase, B = segregation, C = repulsive agglomeration, D = attractive agglomeration. (b) Order parameters along the line $Q = (P-1)/2$ (see solid line in Fig. 1a), averaged over 20 runs after a time period of $20\,000(N_1+N_2)$ update steps with $I = 200$ cells and $N_1 = N_2 = 2000$ entities in each of two subpopulations. The theoretically predicted phase transitions at $P = -1$, $P = 0$, and $P = 1$ (see black triangles in Fig. 1a) are clearly visible.

Let us now focus on the general case with arbitrary payoffs and diffusion. While $D_1, D_2 > 0$ increases the threshold for pattern formation in Eq. (5), in Eq. (6) it can reduce the threshold for moderate diffusion, while the threshold will be higher for large diffusion. We are not surprised that sufficiently large diffusion will always give rise to disorder and suppress pattern formation. It is interesting though that a medium level of diffusion may cause pattern formation where the system would otherwise be disordered. Let us focus on the example with $P_{11} = -2$, $P_{12} = 2$, $P_{21} = -2$, and $P_{22} = 1$, where subpopulation 2 is repelled from subpopulation 1 and where the self-interaction within subpopulation 1 is repulsive as well, while the other interactions are attractive. According to conditions (5) and (6), we expect disorder for small fluctuation strengths p_1, p_2. While increasing values of p_2 should be counterproductive, increasing p_1 should be able to produce pattern formation for medium values of p_1.

This *fluctuation-induced self-organization* is, in fact, observed (see Fig. 2a). The entities in subpopulation 2 can agglomerate at sites, where the fluctuations have temporarily reduced the density in subpopulation 1 due to disturbances of its homogeneous distribution. Later on, subpopulation 1 develops a slightly higher concentration at sites where subpopulation 2 clusters. In a similar way, we can have *drift-induced self-organization* for $V_1^0 = V > 0$ and $V_2^0 = 0$ (see Fig. 2b). For $D_1 = D_2 = 0$, the instability condition (6) is then replaced by

$$\rho_1^0\rho_2^0(P_{12}P_{21} - P_{11}P_{22})k^2(\rho_1^0 P_{11} + \rho_2^0 P_{22})^2 > \rho_1^0\rho_2^0 P_{11}P_{22}(V_1^0 - V_2^0)^2 \, , \qquad (8)$$

where k represents the wave number. (Note that the wave length, which is inversely proportional to the respective wave number, is restricted to the values $\lambda = i\Delta x$ in our cellular automaton simulations.)

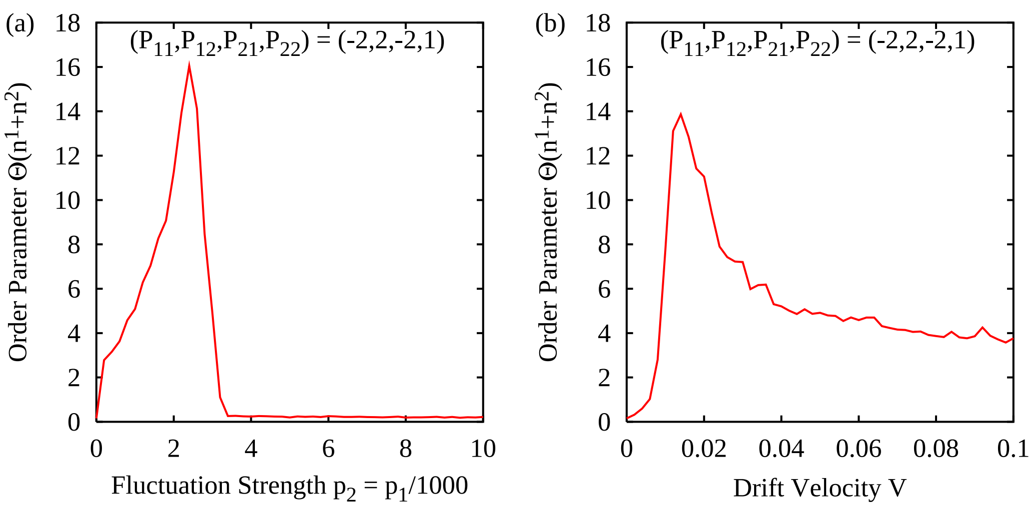

Figure 2: Order parameter for the degree of agglomeration as a function of (a) the fluctuation strength $p_2 = 0.001p_1$ (with $V_1^0 = V_2^0 = 0$) and (b) the drift velocity $V = V_1^0$ (with $V_2^0 = 0 = p_1 = p_2$) for an example with asymmetric interactions. The curves are averages over 20 runs after a time period of $20\,000(N_1 + N_2)$ update steps for $I = 20$ cells and $N_1 = N_2 = 200$ entities in each subpopulation.

For symmetric cases with $P_{11} = P_{22} = P \neq 0$, $P_{12} = P_{21} = Q$, and $\rho_1^0 = \rho_2^0 = \rho \neq 0$, we find $Q^2 - P^2 > (V_1^0 - V_2^0)^2/(2k\rho)^2$, i.e. a finite drift will usually increase the threshold for pattern formation. This is different from the effect of diffusion. For the symmetric case with $V_1^0 = V_2^0 = 0$ and $D_1 = D_2 = D$, the instability condition (6) reads $\rho^2 Q^2 > (\rho P - D)^2$. That is, we expect a "maximum degree of self-organization" for $D = \max(\rho P, 0)$, and a more or less symmetric behavior around this point. What does this actually mean? Fig. 3 suggests that this statement applies to the order in the system. That is, for increasing fluctuations strength we find *noise-induced ordering* up to $D = \max(\rho P, 0)$, while we have disorder at significantly higher fluctuations strengths.

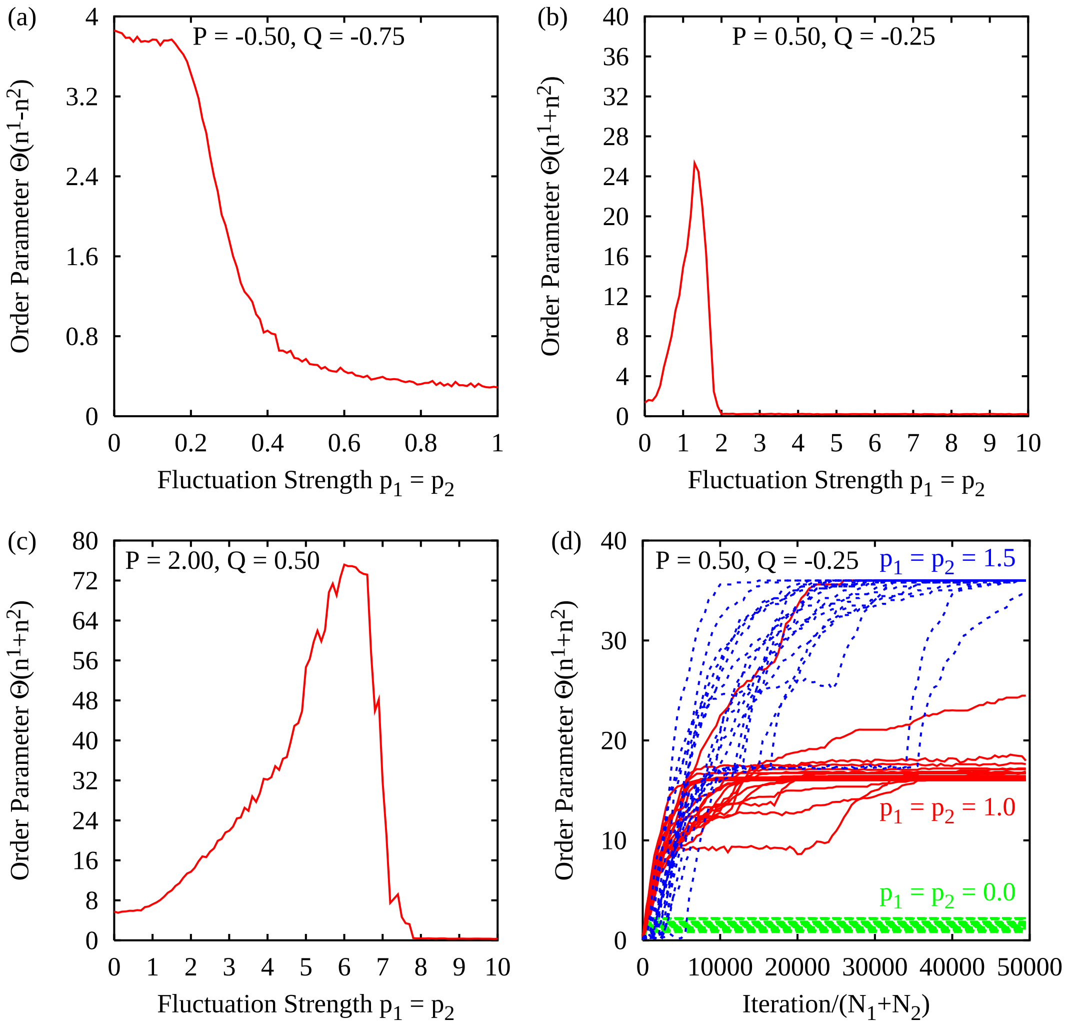

Figure 3: As Fig. 2, but for $V_1^0 = V_2^0 = 0$ and symmetric interactions (see the black circles in Fig. 1a for the location in the parameter space). (a) In cases of segregation, the order stays about constant for small noise amplitudes, but it drops significantly for large ones. In cases of (b) repulsive or (c) attractive agglomeration, a suitable noise amplitude $p_1 = p_2$ can increase the order in the system by more than a factor of 15. The maximum order is reached at $p_1 = p_2 \approx \max(3P, 0)$, corresponding to $D_1 = D_2 = \max(\rho P, 0)$. Close to this maximum, the curves are roughly symmetric, but for less than twice this value, the order breaks down completely. (d) The time-dependence of the order parameter visualizes the increase of the order in the system. The plot shows 20 runs for each displayed fluctuation strength, which significantly influences the dynamics of the ordering process. After large enough times, we find a typical, noise-dependent length scale and level of order in the system.

One may think that this fluctuation-induced ordering is due to noise-induced transitions from a metastable state (local optimum) to a more stable state of higher order (possibly the global optimum). We check this hypothesis for the case of repulsive agglomeration, where a coarse-graining appears particularly difficult because of the repulsion effect. Our observations are as follows: (i) At moderate, but sufficiently large fluctuation strengths, we sometimes observe a "step-wise" fusion of agglomerations, which is associated with exponential-like relaxation processes of the time-dependent order parameter to a higher level. (ii) However, the main mechanism seems to be that, from the very beginning, fluctuations further the formation of larger agglomerations (i.e. suppress the development of small ones), which slows down the ordering process in the early stage. (Sometimes the system stays disordered for more than $500\,000(N_1+N_2)$ iterations and, suddenly, the order increases rapidly to a high level). (iii) A careful choice of the noise strength can speed up the time-dependent increase of the order very much. (iv) After a given, large enough time period, the system has reached a typical level of order, which depends significantly on the fluctuation strength. In conclusion, one may influence the typical length scale in the system by variation of the noise level.

A variation of the "applied" drift velocity (if possible) or of the fluctuation strength together with a proper choice of the "treatment times" would allow one to control pattern formation in several respects: (i) the speed of ordering, (ii) the typical length scale in the system, and (iii) the level of ordering. A time-dependent variation of the control parameters should even facilitate to switch between supporting and suppressing structure formation, e.g. between demixing and homogenization. In the future, these points may, for example, be relevant for the production, properties, handling, and transport of heterogeneous materials, for flow control [20], and efficient separation techniques for different kinds of particles. Due to the general, game-theoretical formulation of the above cellular automaton model, its properties are reminiscent of phenomena in physical, biological and socio-economic systems. For example, we mention phenomena such as the formation of pedestrian lanes (segregation) [17, 18], of ghettos in cities (repulsive agglomeration), or of settlements (attractive agglomeration) [21].

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